Chapter 2

Introduction to Optimization and Linear Programming

Ragsdale, Cliff T.
Spreadsheet Modeling & Decision Analysis, 5e
South-Western, Mason, OH, 2008:17-44
ISBN 978-0-324-65663-3

2.0 Introduction

Our world is filled with limited resources. The amount of oil we can pump out of the earth is limited. The amount of land available for garbage dumps and hazardous waste is limited and, in many areas, diminishing rapidly. On a more personal level, each of us has a limited amount of time in which to accomplish or enjoy the activities we schedule each day. Most of us have a limited amount of money to spend while pursuing these activities. Businesses also have limited resources. A manufacturing organization employs a limited number of workers. A restaurant has a limited amount of space available for seating.

Deciding how best to use the limited resources available to an individual or a business is a universal problem. In today's competitive business environment, it is increasingly important to make sure that a company's limited resources are used in the most efficient manner possible. Typically, this involves determining how to allocate the resources in such a way as to maximize profits or minimize costs. **Mathematical programming** (MP) is a field of management science that finds the optimal, or most efficient, way of using limited resources to achieve the objectives of an individual or a business. For this reason, mathematical programming often is referred to as **optimization**.

2.1 Applications of Mathematical Optimization

To help you understand the purpose of optimization and the types of problems for which it can be used, let's consider several examples of decision-making situations in which MP techniques have been applied.

Determining Product Mix. Most manufacturing companies can make a variety of products. However, each product usually requires different amounts of raw materials and labor. Similarly, the amount of profit generated by the products varies. The manager of such a company must decide how many of each product to produce to maximize profits or to satisfy demand at minimum cost.

Manufacturing. Printed circuit boards, like those used in most computers, often have hundreds or thousands of holes drilled in them to accommodate the different electrical components that must be plugged into them. To manufacture these boards, a computer-controlled drilling machine must be programmed to drill in a given location, then move the drill bit to the next location and drill again. This process is repeated hundreds or thousands of times to complete all the holes on a circuit board. Manufacturers of these boards would benefit from determining the drilling order that minimizes the total distance the drill bit must be moved.

Routing and Logistics. Many retail companies have warehouses around the country that are responsible for keeping stores supplied with merchandise to sell. The amount of merchandise available at the warehouses and the amount needed at each store tends to fluctuate, as does the cost of shipping or delivering merchandise from the warehouses to the retail locations. Large amounts of money can be saved by determining the least costly method of transferring merchandise from the warehouses to the stores.

Financial Planning. The federal government requires individuals to begin withdrawing money from individual retirement accounts (IRAs) and other tax-sheltered retirement programs no later than age 70.5. There are various rules that must be followed to avoid paying penalty taxes on these withdrawals. Most individuals want to withdraw their money in a manner that minimizes the amount of taxes they must pay while still obeying the tax laws.

Optimization Is Everywhere

Going to Disney World this summer? Optimization will be your ubiquitous companion, scheduling the crews and planes, pricing the airline tickets and hotel rooms, even helping to set capacities on the theme park rides. If you use Orbitz to book your flights, an optimization engine sifts through millions of options to find the cheapest fares. If you get directions to your hotel from MapQuest, another optimization engine figures out the most direct route. If you ship souvenirs home, an optimization engine tells UPS which truck to put the packages on, exactly where on the truck the packages should go to make them fastest to load and unload, and what route the driver should follow to make his deliveries most efficiently.

(Adapted from: V. Postrel, "Operation Everything," *The Boston Globe*, June 27, 2004.)

2.2 Characteristics of Optimization Problems

These examples represent just a few areas in which MP has been used successfully. We will consider many other examples throughout this book. However, these examples give you some idea of the issues involved in optimization. For instance, each example involves one or more *decisions* that must be made: How many of each product should be produced? Which hole should be drilled next? How much of each product should be shipped from each warehouse to the various retail locations? How much money should an individual withdraw each year from various retirement accounts?

Also, in each example, restrictions, or *constraints*, are likely to be placed on the alternatives available to the decision maker. In the first example, when determining the number of products to manufacture, a production manager probably is faced with a limited amount of raw materials and a limited amount of labor. In the second example, the drill never should return to a position where a hole has already been drilled. In the third example, there is a physical limitation on the amount of merchandise a truck can carry from one warehouse to the stores on its route. In the fourth example, laws determine the minimum and maximum amounts that can be withdrawn from retirement accounts without incurring a penalty. There might be many other constraints for these examples. Indeed, it is not unusual for real-world optimization problems to have hundreds or thousands of constraints.

A final common element in each of the examples is the existence of some goal or *objective* that the decision maker considers when deciding which course of action is best. In the first example, the production manager can decide to produce several different product mixes given the available resources, but the manager probably will choose the mix of products that maximizes profits. In the second example, a large number of possible drilling patterns can be used, but the ideal pattern probably will involve moving the drill bit the shortest total distance. In the third example, there are numerous ways merchandise can be shipped from the warehouses to supply the stores, but the company probably will want to identify the routing that minimizes the total transportation cost. Finally, in the fourth example, individuals can withdraw money from their retirement accounts in many ways without violating the tax laws, but they probably want to find the method that minimizes their tax liability.

2.3 Expressing Optimization Problems Mathematically

From the preceding discussion, we know that optimization problems involve three elements: decisions, constraints, and an objective. If we intend to build a mathematical model of an optimization problem, we will need mathematical terms or symbols to represent each of these three elements.

2.3.1 DECISIONS

The decisions in an optimization problem often are represented in a mathematical model by the symbols $X_1, X_2, \ldots, X_n$. We will refer to $X_1, X_2, \ldots, X_n$ as the **decision variables** (or simply the variables) in the model. These variables might represent the quantities of different products the production manager can choose to produce. They might represent the amount of different pieces of merchandise to ship from a warehouse to a certain store. They might represent the amount of money to be withdrawn from different retirement accounts.

The exact symbols used to represent the decision variables are not particularly important. You could use $Z_1, Z_2, \ldots, Z_n$ or symbols like Dog, Cat, and Monkey to represent the decision variables in the model. The choice of which symbols to use is largely a matter of personal preference and might vary from one problem to the next.

2.3.2 CONSTRAINTS

The constraints in an optimization problem can be represented in a mathematical model in several ways. Three general ways of expressing the possible constraint relationships in an optimization problem are:

A "less than or equal to" constraint:	$f(X_1, X_2, \ldots, X_n) \leq b$
A "greater than or equal to" constraint:	$f(X_1, X_2, \ldots, X_n) \geq b$
An "equal to" constraint:	$f(X_1, X_2, \ldots, X_n) = b$

In each case, the **constraint** is some function of the decision variables that must be less than or equal to, greater than or equal to, or equal to some specific value (represented above by the letter b). We will refer to $f(X_1, X_2, \ldots, X_n)$ as the left-hand-side (LHS) of the constraint and to b as the right-hand-side (RHS) value of the constraint.

For example, we might use a "less than or equal to" constraint to ensure that the total labor used in producing a given number of products does not exceed the amount of available labor. We might use a "greater than or equal to" constraint to ensure that the total amount of money withdrawn from a person's retirement accounts is at least the minimum amount required by the IRS. You can use any number of these constraints to represent a given optimization problem depending on the requirements of the situation.

2.3.3 OBJECTIVE

The objective in an optimization problem is represented mathematically by an objective function in the general format:

$$\text{MAX (or MIN):} \quad f(X_1, X_2, \ldots, X_n)$$

The **objective function** identifies some function of the decision variables that the decision maker wants to either MAXimize or MINimize. In our earlier examples, this function might be used to describe the total profit associated with a product mix, the total distance the drill bit must be moved, the total cost of transporting merchandise, or a retiree's total tax liability.

The mathematical formulation of an optimization problem can be described in the general format:

$$\text{MAX (or MIN):} \quad f_0(X_1, X_2, \ldots, X_n) \qquad \textbf{2.1}$$

$$\text{Subject to:} \quad f_1(X_1, X_2, \ldots, X_n) \leq b_1 \qquad \textbf{2.2}$$

$$\vdots$$

$$f_k(X_1, X_2, \ldots, X_n) \geq b_k \qquad \textbf{2.3}$$

$$\vdots$$

$$f_m(X_1, X_2, \ldots, X_n) = b_m \qquad \textbf{2.4}$$

This representation identifies the objective function (equation 2.1) that will be maximized (or minimized) and the constraints that must be satisfied (equations 2.2 through 2.4). Subscripts added to the f and b in each equation emphasize that the functions describing the objective and constraints can all be different. The goal in optimization is to find the values of the decision variables that maximize (or minimize) the objective function without violating any of the constraints.

2.4 Mathematical Programming Techniques

Our general representation of an MP model is just that—general. You can use many kinds of functions to represent the objective function and the constraints in an MP model. Of course, you always should use functions that accurately describe the objective and constraints of the problem you are trying to solve. Sometimes, the functions in a model are linear in nature (that is, they form straight lines or flat surfaces); other times, they are nonlinear (that is, they form curved lines or curved surfaces). Sometimes, the optimal values of the decision variables in a model must take on integer values (whole numbers); other times, the decision variables can assume fractional values.

Given the diversity of MP problems that can be encountered, many techniques have been developed to solve different types of MP problems. In the next several chapters, we will look at these MP techniques and develop an understanding of how they differ and when each should be used. We will begin by examining a technique called **linear programming** (LP), which involves creating and solving optimization problems with linear objective functions and linear constraints. LP is a very powerful tool that can be applied in many business situations. It also forms a basis for several other techniques discussed later and is, therefore, a good starting point for our investigation into the field of optimization.

2.5 An Example LP Problem

We will begin our study of LP by considering a simple example. You should not interpret this to mean that LP cannot solve more complex or realistic problems. LP has been used to solve extremely complicated problems, saving companies millions of dollars. However, jumping directly into one of these complicated problems would be like starting a marathon without ever having gone out for a jog—you would get winded and could be left behind very quickly. So we'll start with something simple.

> Blue Ridge Hot Tubs manufactures and sells two models of hot tubs: the Aqua-Spa and the Hydro-Lux. Howie Jones, the owner and manager of the company, needs to decide how many of each type of hot tub to produce during his next production cycle. Howie buys prefabricated fiberglass hot tub shells from a local supplier and adds the pump and tubing to the shells to create his hot tubs. (This supplier has the capacity to deliver as many hot tub shells as Howie needs.) Howie installs the same type of pump into both hot tubs. He will have only 200 pumps available during his next production cycle. From a manufacturing standpoint, the main difference between the two models of hot tubs is the amount of tubing and labor required. Each Aqua-Spa requires 9 hours of labor and 12 feet of tubing. Each Hydro-Lux requires 6 hours of labor and 16 feet of tubing. Howie expects to have 1,566 production labor hours and 2,880 feet of tubing available during the next production cycle. Howie earns a profit of $350 on each Aqua-Spa he sells and $300 on each Hydro-Lux he sells. He is confident that he can sell all the hot tubs he produces. The question is, how many Aqua-Spas and Hydro-Luxes should Howie produce if he wants to maximize his profits during the next production cycle?

2.6 Formulating LP Models

The process of taking a practical problem—such as determining how many Aqua-Spas and Hydro-Luxes Howie should produce—and expressing it algebraically in the form of an LP model is known as **formulating** the model. Throughout the next several chapters, you will see that formulating an LP model is as much an art as a science.

2.6.1 STEPS IN FORMULATING AN LP MODEL

There are some general steps you can follow to help make sure your formulation of a particular problem is accurate. We will walk through these steps using the hot tub example.

1. **Understand the problem.** This step appears to be so obvious that it hardly seems worth mentioning. However, many people have a tendency to jump into a problem and start writing the objective function and constraints before they really understand the problem. If you do not fully understand the problem you face, it is unlikely that your formulation of the problem will be correct.

   The problem in our example is fairly easy to understand: How many Aqua-Spas and Hydro-Luxes should Howie produce to maximize his profit, while using no more than 200 pumps, 1,566 labor hours, and 2,880 feet of tubing?

2. **Identify the decision variables.** After you are sure you understand the problem, you need to identify the decision variables. That is, what are the fundamental decisions that must be made to solve the problem? The answers to this question often will help you identify appropriate decision variables for your model. Identifying the decision variables means determining what the symbols $X_1, X_2, \ldots, X_n$ represent in your model.

   In our example, the fundamental decision Howie faces is this: How many Aqua-Spas and Hydro-Luxes should be produced? In this problem, we will let X_1 represent the number of Aqua-Spas to produce and X_2 represent the number of Hydro-Luxes to produce.

3. **State the objective function as a linear combination of the decision variables.** After determining the decision variables you will use, the next step is to create the objective function for the model. This function expresses the mathematical relationship between the decision variables in the model to be maximized or minimized.

   In our example, Howie earns a profit of \$350 on each Aqua-Spa (X_1) he sells and \$300 on each Hydro-Lux (X_2) he sells. Thus, Howie's objective of maximizing the profit he earns is stated mathematically as:

$$\text{MAX:} \quad 350X_1 + 300X_2$$

   For whatever values might be assigned to X_1 and X_2, the previous function calculates the associated total profit that Howie would earn. Obviously, Howie wants to maximize this value.

4. **State the constraints as linear combinations of the decision variables.** As mentioned earlier, there are usually some limitations on the values that can be assumed by the decision variables in an LP model. These restrictions must be identified and stated in the form of constraints.

   In our example, Howie faces three major constraints. Because only 200 pumps are available and each hot tub requires one pump, Howie cannot produce more than a total of 200 hot tubs. This restriction is stated mathematically as:

$$1X_1 + 1X_2 \leq 200$$

   This constraint indicates that each unit of X_1 produced (that is, each Aqua-Spa built) will use one of the 200 pumps available—as will each unit of X_2 produced (that is, each Hydro-Lux built). The total number of pumps used (represented by $1X_1 + 1X_2$) must be less than or equal to 200.

   Another restriction Howie faces is that he has only 1,566 labor hours available during the next production cycle. Because each Aqua-Spa he builds (each unit of X_1) requires 9 labor hours and each Hydro-Lux (each unit of X_2) requires 6 labor hours, the constraint on the number of labor hours is stated as:

$$9X_1 + 6X_2 \leq 1{,}566$$

The total number of labor hours used (represented by $9X_1 + 6X_2$) must be less than or equal to the total labor hours available, which is 1,566.

The final constraint specifies that only 2,880 feet of tubing is available for the next production cycle. Each Aqua-Spa produced (each unit of X_1) requires 12 feet of tubing, and each Hydro-Lux produced (each unit of X_2) requires 16 feet of tubing. The following constraint is necessary to ensure that Howie's production plan does not use more tubing than is available:

$$12X_1 + 16X_2 \leq 2{,}880$$

The total number of feet of tubing used (represented by $12X_1 + 16X_2$) must be less than or equal to the total number of feet of tubing available, which is 2,880.

5. **Identify any upper or lower bounds on the decision variables.** Often, simple upper or lower bounds apply to the decision variables. You can view upper and lower bounds as additional constraints in the problem.

   In our example, there are simple lower bounds of zero on the variables X_1 and X_2 because it is impossible to produce a negative number of hot tubs. Therefore, the following two constraints also apply to this problem:

$$X_1 \geq 0$$
$$X_2 \geq 0$$

   Constraints like these are often referred to as nonnegativity conditions and are quite common in LP problems.

2.7 Summary of the LP Model for the Example Problem

The complete LP model for Howie's decision problem can be stated as:

MAX:	$350X_1 + 300X_2$	**2.5**
Subject to:	$1X_1 + 1X_2 \leq 200$	**2.6**
	$9X_1 + 6X_2 \leq 1{,}566$	**2.7**
	$12X_1 + 16X_2 \leq 2{,}880$	**2.8**
	$1X_1 \geq 0$	**2.9**
	$1X_2 \geq 0$	**2.10**

In this model, the decision variables X_1 and X_2 represent the number of Aqua-Spas and Hydro-Luxes to produce, respectively. Our goal is to determine the values for X_1 and X_2 that maximize the objective in equation 2.5 while simultaneously satisfying all the constraints in equations 2.6 through 2.10.

2.8 The General Form of an LP Model

The technique of linear programming is so named because the MP problems to which it applies are linear in nature. That is, it must be possible to express all the functions in an

LP model as some weighted sum (or linear combination) of the decision variables. So, an LP model takes on the general form:

$$\text{MAX (or MIN):} \quad c_1X_1 + c_2X_2 + \cdots + c_nX_n \tag{2.11}$$

$$\text{Subject to:} \quad a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \leq b_1 \tag{2.12}$$

$$\vdots$$

$$a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n \geq b_k \tag{2.13}$$

$$\vdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m \tag{2.14}$$

Up to this point, we have suggested that the constraints in an LP model represent some type of limited resource. Although this is frequently the case, in later chapters you will see examples of LP models in which the constraints represent things other than limited resources. The important point here is that *any* problem that can be formulated in the above fashion is an LP problem.

The symbols $c_1, c_2, \ldots, c_n$ in equation 2.11 are called **objective function coefficients** and might represent the marginal profits (or costs) associated with the decision variables $X_1, X_2, \ldots, X_n$, respectively. The symbol a_{ij} found throughout equations 2.12 through 2.14 represents the numeric coefficient in the *i*th constraint for variable X_j. The objective function and constraints of an LP problem represent different weighted sums of the decision variables. The b_i symbols in the constraints, once again, represent values that the corresponding linear combination of the decision variables must be less than or equal to, greater than or equal to, or equal to.

You should now see a direct connection between the LP model we formulated for Blue Ridge Hot Tubs in equations 2.5 through 2.10 and the general definition of an LP model given in equations 2.11 through 2.14. In particular, note that the various symbols used in equations 2.11 through 2.14 to represent numeric constants (that is, the c_j, a_{ij}, and b_i) were replaced by actual numeric values in equations 2.5 through 2.10. Also, note that our formulation of the LP model for Blue Ridge Hot Tubs did not require the use of "equal to" constraints. Different problems require different types of constraints, and you should use whatever types of constraints are necessary for the problem at hand.

2.9 Solving LP Problems: An Intuitive Approach

After an LP model has been formulated, our interest naturally turns to solving it. But before we actually solve our example problem for Blue Ridge Hot Tubs, what do you think is the optimal solution to the problem? Just by looking at the model, what values for X_1 and X_2 do you think would give Howie the largest profit?

Following one line of reasoning, it might seem that Howie should produce as many units of X_1 (Aqua-Spas) as possible because each of these generates a profit of $350, whereas each unit of X_2 (Hydro-Luxes) generates a profit of only $300. But what is the maximum number of Aqua-Spas that Howie could produce?

Howie can produce the maximum number of units of X_1 by making no units of X_2 and devoting all his resources to the production of X_1. Suppose we let $X_2 = 0$ in the model in equations 2.5 through 2.10 to indicate that no Hydro-Luxes will be produced.

What then is the largest possible value of X_1? If $X_2 = 0$, then the inequality in equation 2.6 tells us:

$$X_1 \leq 200 \quad \textbf{2.15}$$

So we know that X_1 cannot be any greater than 200 if $X_2 = 0$. However, we also have to consider the constraints in equations 2.7 and 2.8. If $X_2 = 0$, then the inequality in equation 2.7 reduces to:

$$9X_1 \leq 1{,}566 \quad \textbf{2.16}$$

If we divide both sides of this inequality by 9, we find that the previous constraint is equivalent to:

$$X_1 \leq 174 \quad \textbf{2.17}$$

Now consider the constraint in equation 2.8. If $X_2 = 0$, then the inequality in equation 2.8 reduces to:

$$12X_1 \leq 2{,}880 \quad \textbf{2.18}$$

Again, if we divide both sides of this inequality by 12, we find that the previous constraint is equivalent to:

$$X_1 \leq 240 \quad \textbf{2.19}$$

So, if $X_2 = 0$, the three constraints in our model imposing upper limits on the value of X_1 reduce to the values shown in equations 2.15, 2.17, and 2.19. The most restrictive of these constraints is equation 2.17. Therefore, the maximum number of units of X_1 that can be produced is 174. In other words, 174 is the largest value X_1 can take on and still satisfy all the constraints in the model.

If Howie builds 174 units of X_1 (Aqua-Spas) and 0 units of X_2 (Hydro-Luxes), he will have used all of the labor that is available for production ($9X_1 = 1{,}566$ if $X_1 = 174$). However, he will have 26 pumps remaining ($200 - X_1 = 26$ if $X_1 = 174$) and 792 feet of tubing remaining ($2{,}880 - 12X_1 = 792$ if $X_1 = 174$). Also, notice that the objective function value (or total profit) associated with this solution is:

$$\$350X_1 + \$300X_2 = \$350 \times 174 + \$300 \times 0 = \$60{,}900$$

From this analysis, we see that the solution $X_1 = 174$, $X_2 = 0$ is a *feasible solution* to the problem because it satisfies all the constraints of the model. But is it the *optimal solution*? In other words, is there any other possible set of values for X_1 and X_2 that also satisfies all the constraints *and* results in a higher objective function value? As you will see, the intuitive approach to solving LP problems that we have taken here cannot be trusted because there actually is a *better* solution to Howie's problem.

2.10 Solving LP Problems: A Graphical Approach

The constraints of an LP model define the set of feasible solutions—or the **feasible region**—for the problem. The difficulty in LP is determining which point or points in the feasible region correspond to the best possible value of the objective function. For simple problems with only two decision variables, it is fairly easy to sketch the feasible region for the LP model and locate the optimal feasible point graphically. Because the graphical approach can be used only if there are two decision variables, it has limited practical use. However, it is an extremely good way to develop a basic understanding of the strategy involved in solving LP problems. Therefore, we will use the graphical approach to solve the simple problem faced by Blue Ridge Hot Tubs. Chapter 3 shows how to solve this and other LP problems using a spreadsheet.

To solve an LP problem graphically, first you must plot the constraints for the problem and identify its feasible region. This is done by plotting the *boundary lines* of the constraints and identifying the points that will satisfy all the constraints. So, how do we do this for our example problem (repeated below)?

MAX:	$350X_1 + 300X_2$	**2.20**
Subject to:	$1X_1 + 1X_2 \leq 200$	**2.21**
	$9X_1 + 6X_2 \leq 1{,}566$	**2.22**
	$12X_1 + 16X_2 \leq 2{,}880$	**2.23**
	$1X_1 \geq 0$	**2.24**
	$1X_2 \geq 0$	**2.25**

2.10.1 PLOTTING THE FIRST CONSTRAINT

The boundary of the first constraint in our model, which specifies that no more than 200 pumps can be used, is represented by the straight line defined by the equation:

$$X_1 + X_2 = 200 \qquad \textbf{2.26}$$

If we can find any two points on this line, the entire line can be plotted easily by drawing a straight line through these points. If $X_2 = 0$, we can see from equation 2.26 that $X_1 = 200$. Thus, the point $(X_1, X_2) = (200, 0)$ must fall on this line. If we let $X_1 = 0$, from equation 2.26, it is easy to see that $X_2 = 200$. So, the point $(X_1, X_2) = (0, 200)$ also must fall on this line. These two points are plotted on the graph in Figure 2.1 and connected to form the straight line representing equation 2.26.

Note that the graph of the line associated with equation 2.26 actually extends beyond the X_1 and X_2 axes shown in Figure 2.1. However, we can disregard the points beyond these axes because the values assumed by X_1 and X_2 cannot be negative (because we also have the constraints given by $X_1 \geq 0$ and $X_2 \geq 0$).

The line connecting the points (0, 200) and (200, 0) in Figure 2.1 identifies the points (X_1, X_2) that satisfy the equality $X_1 + X_2 = 200$. But recall that the first constraint in the LP model is the inequality $X_1 + X_2 \leq 200$. Thus, after plotting the boundary line of a constraint, we must determine which area on the graph corresponds to feasible solutions for the original constraint. This can be done easily by picking an arbitrary point on either side of (*i.e.*, not on) the boundary line and checking whether it satisfies the original constraint. For example, the point $(X_1, X_2) = (0, 0)$ is not on the boundary line of the first constraint and also satisfies the first constraint. Therefore, the area of the graph on the same side of the boundary line as the point (0, 0) corresponds to the feasible solutions of our first constraint. This area of feasible solutions is shaded in Figure 2.1.

2.10.2 PLOTTING THE SECOND CONSTRAINT

Some of the feasible solutions to one constraint in an LP model usually will not satisfy one or more of the other constraints in the model. For example, the point $(X_1, X_2) = (200, 0)$ satisfies the first constraint in our model, but it does not satisfy the second constraint, which requires that no more than 1,566 labor hours be used (because $9 \times 200 + 6 \times 0 = 1{,}800$). So, what values for X_1 and X_2 will satisfy both of these

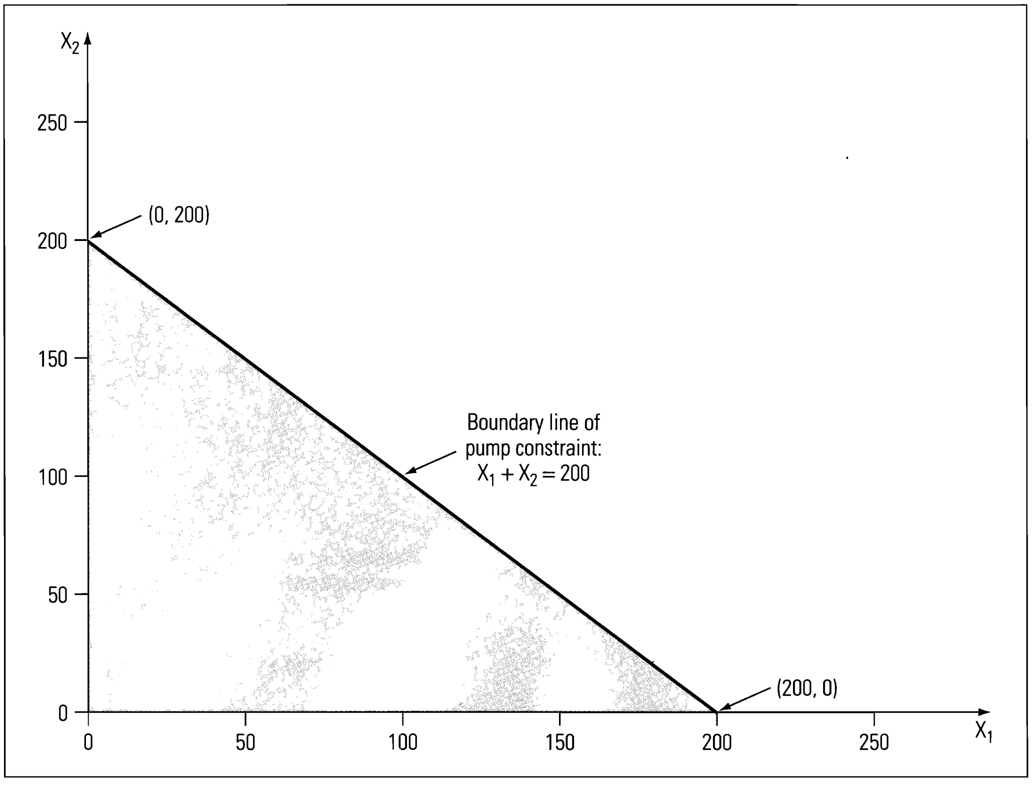


FIGURE 2.1

Graphical representation of the pump constraint

constraints simultaneously? To answer this question, we also need to plot the second constraint on the graph. This is done in the same manner as before—by locating two points on the boundary line of the constraint and connecting these points with a straight line.

The boundary line for the second constraint in our model is given by:

$$9X_1 + 6X_2 = 1{,}566 \qquad \textbf{2.27}$$

If $X_1 = 0$ in equation 2.27, then $X_2 = 1{,}566/6 = 261$. So, the point (0, 261) must fall on the line defined by equation 2.27. Similarly, if $X_2 = 0$ in equation 2.27, then $X_1 = 1{,}566/9 = 174$. So, the point (174, 0) also must fall on this line. These two points are plotted on the graph and connected with a straight line representing equation 2.27, as shown in Figure 2.2.

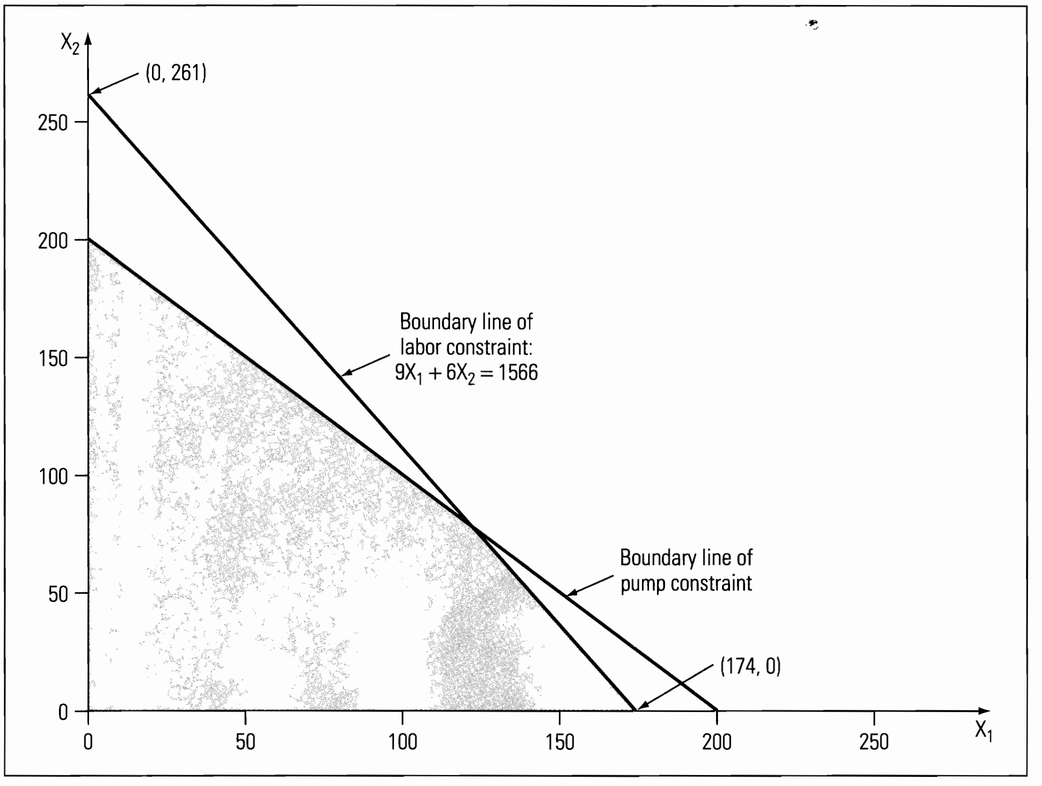
The line drawn in Figure 2.2 representing equation 2.27 is the boundary line for our second constraint. To determine the area on the graph that corresponds to feasible solutions to the second constraint, we again need to test a point on either side of this line to see if it is feasible. The point $(X_1, X_2) = (0, 0)$ satisfies $9X_1 + 6X_2 \leq 1{,}566$. Therefore, all points on the same side of the boundary line satisfy this constraint.

2.10.3 PLOTTING THE THIRD CONSTRAINT

To find the set of values for X_1 and X_2 that satisfies all the constraints in the model, we need to plot the third constraint. This constraint requires that no more than 2,880 feet of tubing be used in producing the hot tubs. Again, we will find two points on the graph that fall on the boundary line for this constraint and connect them with a straight line.

FIGURE 2.2

Graphical representation of the pump and labor constraints



The boundary line for the third constraint in our model is:

$$12X_1 + 16X_2 = 2{,}880 \qquad \textbf{2.28}$$

If $X_1 = 0$ in equation 2.28, then $X_2 = 2{,}880/16 = 180$. So, the point (0, 180) must fall on the line defined by equation 2.28. Similarly, if $X_2 = 0$ in equation 2.28, then $X_1 = 2{,}880/12 = 240$. So, the point (240, 0) also must fall on this line. These two points are plotted on the graph and connected with a straight line representing equation 2.28, as shown in Figure 2.3.

Again, the line drawn in Figure 2.3 representing equation 2.28 is the boundary line for our third constraint. To determine the area on the graph that corresponds to feasible solutions to this constraint, we need to test a point on either side of this line to see if it is feasible. The point $(X_1, X_2) = (0, 0)$ satisfies $12X_1 + 16X_2 \le 2{,}880$. Therefore, all points on the same side of the boundary line satisfy this constraint.

2.10.4 THE FEASIBLE REGION

It is now easy to see which points satisfy all the constraints in our model. These points correspond to the shaded area in Figure 2.3, labeled "Feasible Region." The **feasible region** is the set of points or values that the decision variables can assume and simultaneously satisfy all the constraints in the problem. Take a moment now to carefully compare the graphs in Figures 2.1, 2.2, and 2.3. In particular, notice that when we added the second constraint in Figure 2.2, some of the feasible solutions associated with the first constraint were eliminated because these solutions did not satisfy the second constraint. Similarly, when we added the third constraint in Figure 2.3, another portion of the feasible solutions for the first constraint was eliminated.

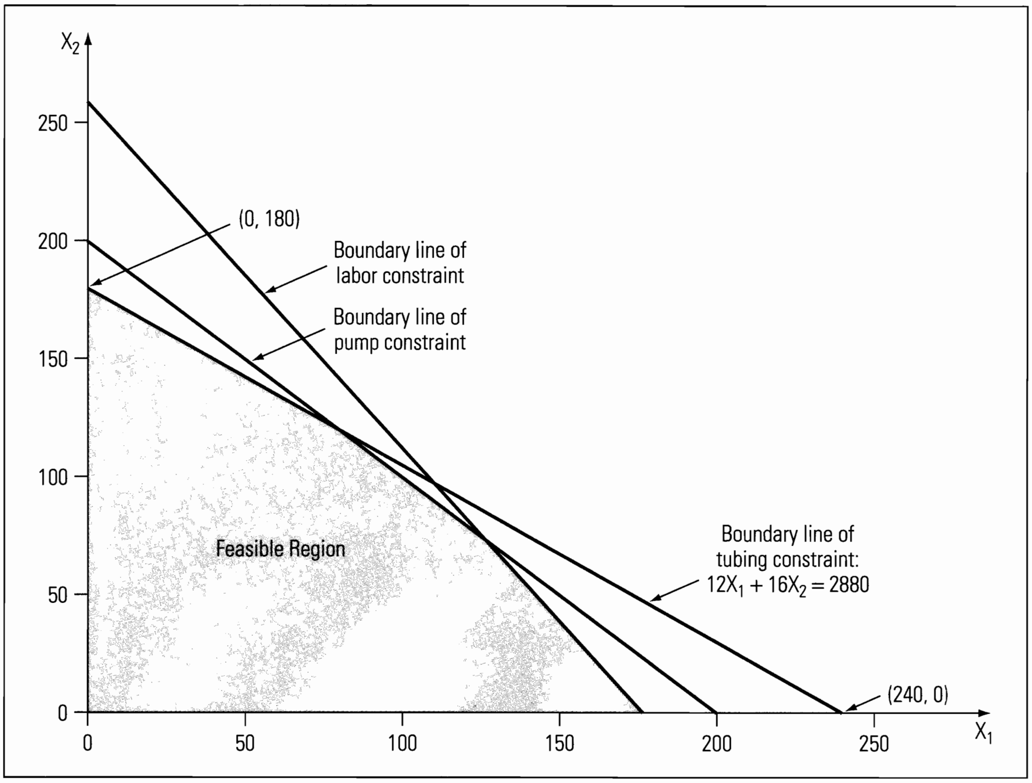


FIGURE 2.3

Graphical representation of the feasible region

2.10.5 PLOTTING THE OBJECTIVE FUNCTION

Now that we have isolated the set of feasible solutions to our LP problem, we need to determine which of these solutions is best. That is, we must determine which point in the feasible region will maximize the value of the objective function in our model. At first glance, it might seem that trying to locate this point is like searching for a needle in a haystack. After all, as shown by the shaded region in Figure 2.3, there are an *infinite* number of feasible solutions to this problem. Fortunately, it is easy to eliminate most of the feasible solutions in an LP problem from consideration. It can be shown that if an LP problem has an optimal solution with a finite objective function value, this solution always will occur at a point in the feasible region where two or more of the boundary lines of the constraints intersect. These points of intersection are sometimes called **corner points** or **extreme points** of the feasible region.

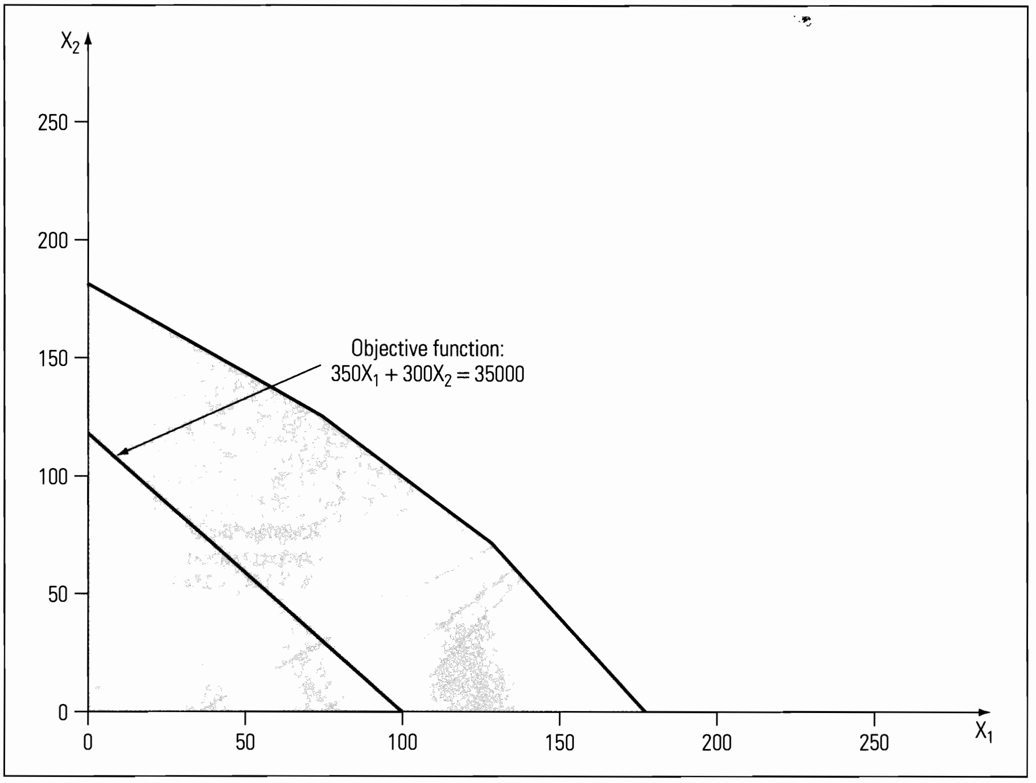
To see why the finite optimal solution to an LP problem occurs at an extreme point of the feasible region, consider the relationship between the objective function and the feasible region of our example LP model. Suppose we are interested in finding the values of X_1 and X_2 associated with a given level of profit, such as \$35,000. Then, mathematically, we are interested in finding the points (X_1, X_2) for which our objective function equals \$35,000, or where:

$$\$350X_1 + \$300X_2 = \$35{,}000 \qquad \textbf{2.29}$$

This equation defines a straight line, which we can plot on our graph. Specifically, if $X_1 = 0$ then, from equation 2.29, $X_2 = 116.67$. Similarly, if $X_2 = 0$ in equation 2.29, then $X_1 = 100$. So, the points $(X_1, X_2) = (0, 116.67)$ and $(X_1, X_2) = (100, 0)$ both fall on the line defining a profit level of \$35,000. (Note that all the points on this line produce a profit level of \$35,000.) This line is shown in Figure 2.4.

FIGURE 2.4

Graph showing values of X_1 and X_2 that produce an objective function value of \$35,000



Now, suppose we are interested in finding the values of X_1 and X_2 that produce some higher level of profit, such as \$52,500. Then, mathematically, we are interested in finding the points (X_1, X_2) for which our objective function equals \$52,500, or where:

$$\$350X_1 + \$300X_2 = \$52{,}500 \qquad \textbf{2.30}$$

This equation also defines a straight line, which we could plot on our graph. If we do this, we'll find that the points (X_1, X_2) = (0, 175) and (X_1, X_2) = (150, 0) both fall on this line, as shown in Figure 2.5.

2.10.6 FINDING THE OPTIMAL SOLUTION USING LEVEL CURVES

The lines in Figure 2.5 representing the two objective function values are sometimes referred to as **level curves** because they represent different levels or values of the objective. Note that the two level curves in Figure 2.5 are *parallel* to one another. If we repeat this process of drawing lines associated with larger and larger values of our objective function, we will continue to observe a series of parallel lines shifting away from the origin—that is, away from the point (0, 0). The very last level curve we can draw that still intersects the feasible region will determine the maximum profit we can achieve. This point of intersection, shown in Figure 2.6, represents the optimal feasible solution to the problem.

As shown in Figure 2.6, the optimal solution to our example problem occurs at the point where the largest possible level curve intersects the feasible region at a single point. This is the feasible point that produces the largest profit for Blue Ridge Hot Tubs. But how do we figure out exactly what point this is and how much profit it provides?

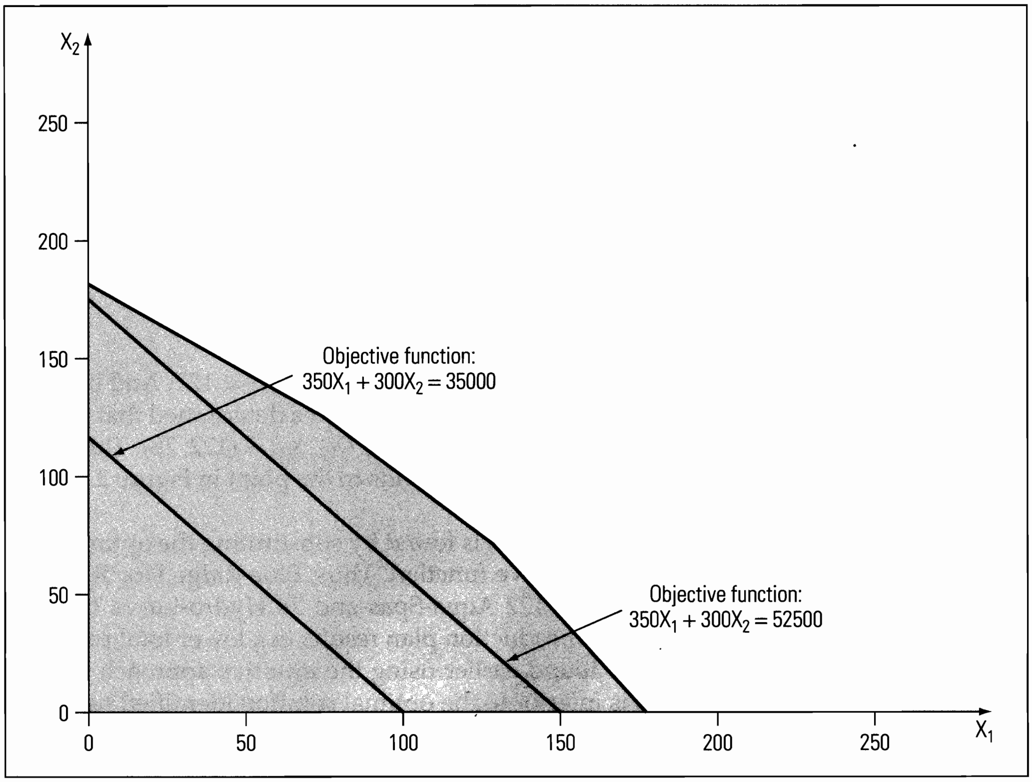


FIGURE 2.5

Parallel level curves for two different objective function values

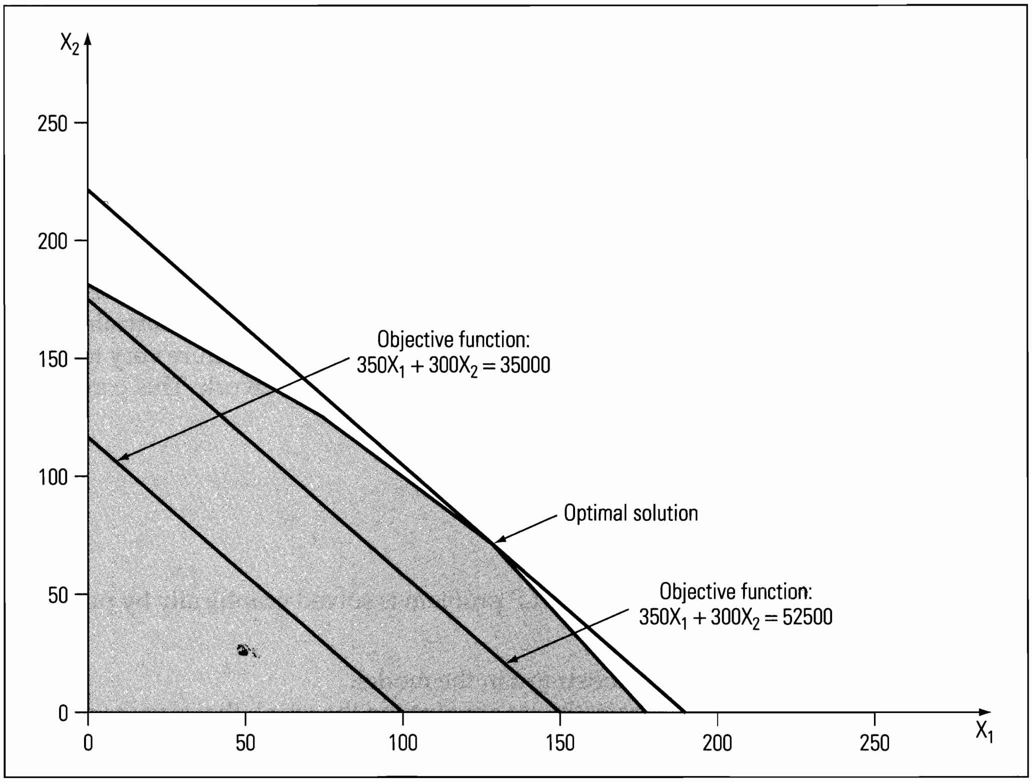


FIGURE 2.6

Graph showing optimal solution where the level curve is tangent to the feasible region

If you compare Figure 2.6 to Figure 2.3, you see that the optimal solution occurs where the boundary lines of the pump and labor constraints intersect (or are equal). Thus, the optimal solution is defined by the point (X_1, X_2) that simultaneously satisfies equations 2.26 and 2.27, which are repeated below:

$$X_1 + X_2 = 200$$
$$9X_1 + 6X_2 = 1{,}566$$

From the first equation, we easily conclude that $X_2 = 200 - X_1$. If we substitute this definition of X_2 into the second equation we obtain:

$$9X_1 + 6(200 - X_1) = 1{,}566$$

Using simple algebra, we can solve this equation to find that $X_1 = 122$. And because $X_2 = 200 - X_1$, we can conclude that $X_2 = 78$. Therefore, we have determined that the optimal solution to our example problem occurs at the point $(X_1, X_2) = (122, 78)$. This point satisfies all the constraints in our model and corresponds to the point in Figure 2.6 identified as the optimal solution.

The total profit associated with this solution is found by substituting the optimal values of $X_1 = 122$ and $X_2 = 78$ into the objective function. Thus, Blue Ridge Hot Tubs can realize a profit of \$66,100 if it produces 122 Aqua-Spas and 78 Hydro-Luxes (\$350 × 122 + \$300 × 78 = \$66,100). Any other production plan results in a lower total profit. In particular, note that the solution we found earlier using the intuitive approach (which produced a total profit of \$60,900) is inferior to the optimal solution identified here.

2.10.7 FINDING THE OPTIMAL SOLUTION BY ENUMERATING THE CORNER POINTS

Earlier, we indicated that if an LP problem has a finite optimal solution, this solution always will occur at some corner point of the feasible region. So, another way of solving an LP problem is to identify all the corner points, or extreme points, of the feasible region and calculate the value of the objective function at each of these points. The corner point with the largest objective function value is the optimal solution to the problem.

This approach is illustrated in Figure 2.7, where the X_1 and X_2 coordinates for each of the extreme points are identified along with the associated objective function values. As expected, this analysis also indicates that the point $(X_1, X_2) = (122, 78)$ is optimal.

Enumerating the corner points to identify the optimal solution is often more difficult than the level curve approach because it requires that you identify the coordinates for *all* the extreme points of the feasible region. If there are many intersecting constraints, the number of extreme points can become rather large, making this procedure very tedious. Also, a special condition exists for which this procedure will not work. This condition, known as an unbounded solution, is described shortly.

2.10.8 SUMMARY OF GRAPHICAL SOLUTION TO LP PROBLEMS

To summarize this section, a two-variable LP problem is solved graphically by performing these steps:

1. Plot the boundary line of each constraint in the model.
2. Identify the feasible region, that is, the set of points on the graph that simultaneously satisfies all the constraints.

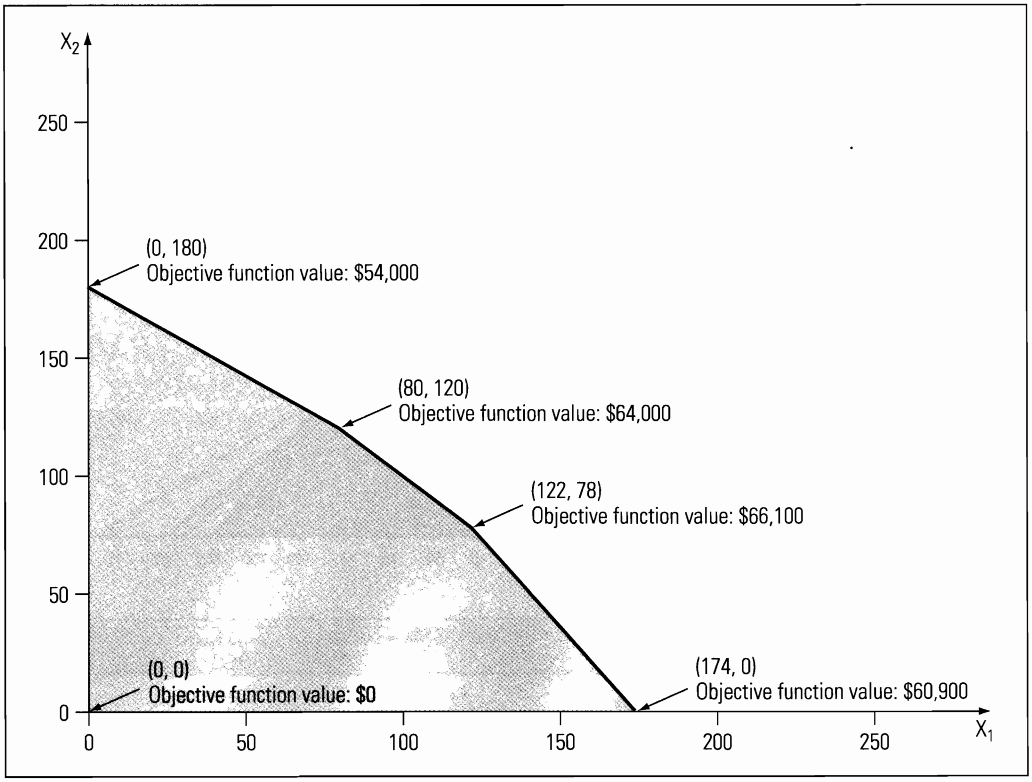


FIGURE 2.7

Objective function values at each extreme point of the feasible region

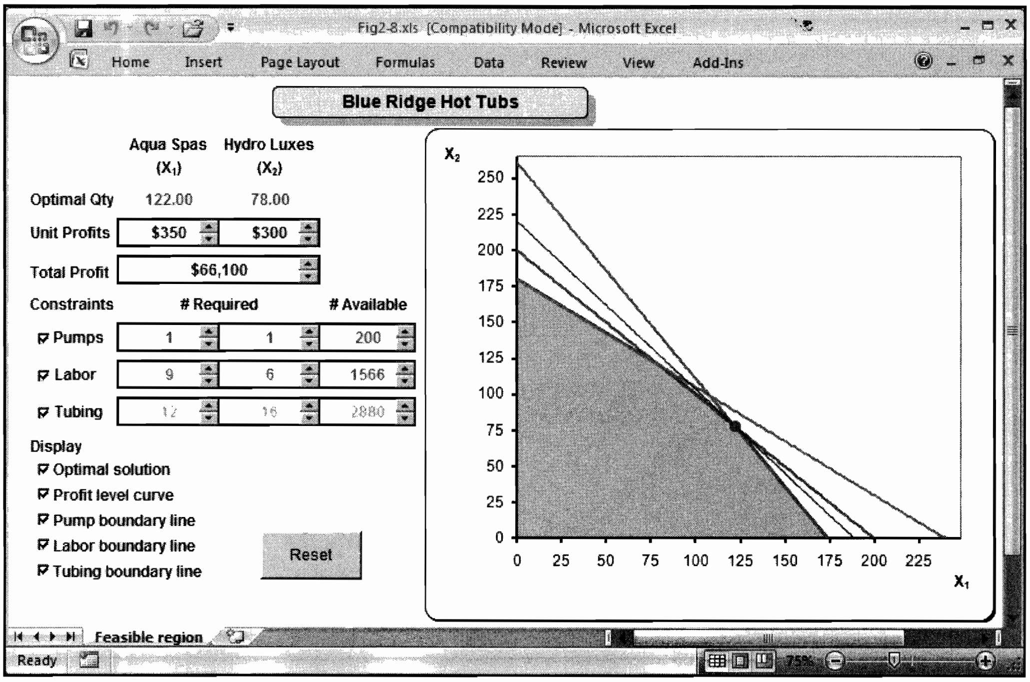
3. Locate the optimal solution by one of the following methods:
   a. Plot one or more level curves for the objective function and determine the direction in which parallel shifts in this line produce improved objective function values. Shift the level curve in a parallel manner in the improving direction until it intersects the feasible region at a single point. Then find the coordinates for this point. This is the optimal solution.
   b. Identify the coordinates of all the extreme points of the feasible region and calculate the objective function values associated with each point. If the feasible region is bounded, the point with the best objective function value is the optimal solution.

2.10.9 UNDERSTANDING HOW THINGS CHANGE

It is important to realize that if changes occur in any of the coefficients in the objective function or constraints of this problem, then the level curve, feasible region, and optimal solution to this problem also might change. To be an effective LP modeler, it is important for you to develop some intuition about how changes in various coefficients in the model will affect the solution to the problem. We will study this in greater detail in Chapter 4 when discussing sensitivity analysis. However, the spreadsheet shown in Figure 2.8 (and in the file named Fig2-8.xls on your data disk) allows you to change any of the coefficients in this problem and, instantly, see its effect. You are encouraged to experiment with this file to make sure that you understand the relationships between various model coefficients and their impact on this LP problem. (Case 2-1 at the end of this chapter asks some specific questions that can be answered using the spreadsheet shown in Figure 2.8.)

FIGURE 2.8

Interactive spreadsheet for the Blue Ridge Hot Tubs LP problem



2.11 Special Conditions in LP Models

Several special conditions can arise in LP modeling: *alternate optimal solutions, redundant constraints, unbounded solutions,* and *infeasibility*. The first two conditions do not prevent you from solving an LP model and are not really problems—they are just anomalies that sometimes occur. On the other hand, the last two conditions represent real problems that prevent us from solving an LP model.

2.11.1 ALTERNATE OPTIMAL SOLUTIONS

Some LP models can have more than one optimal solution, or **alternate optimal solutions**. That is, there can be more than one feasible point that maximizes (or minimizes) the value of the objective function.

For example, suppose Howie can increase the price of Aqua-Spas to the point at which each unit sold generates a profit of $450 rather than $350. The revised LP model for this problem is:

$$\begin{aligned}
\text{MAX:}\quad & 450X_1 + 300X_2 \\
\text{Subject to:}\quad & 1X_1 + 1X_2 \leq 200 \\
& 9X_1 + 6X_2 \leq 1{,}566 \\
& 12X_1 + 16X_2 \leq 2{,}880 \\
& 1X_1 \geq 0 \\
& 1X_2 \geq 0
\end{aligned}$$

Because none of the constraints changed, the feasible region for this model is the same as for the earlier example. The only difference in this model is the objective function. Therefore, the level curves for the objective function are different from what we observed earlier. Several level curves for this model are plotted with its feasible region in Figure 2.9.

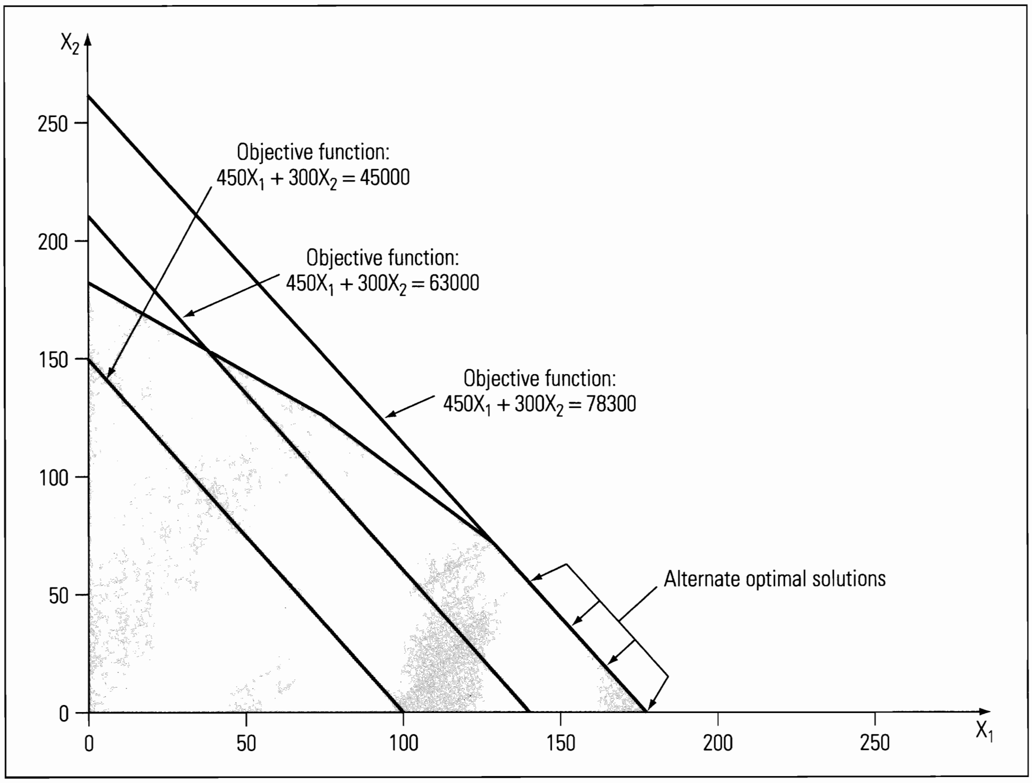


FIGURE 2.9

Example of an LP problem with an infinite number of alternate optimal solutions

Notice that the final level curve in Figure 2.9 intersects the feasible region along an *edge* of the feasible region rather than at a single point. All the points on the line segment joining the corner point at (122, 78) to the corner point at (174, 0) produce the same optimal objective function value of $78,300 for this problem. Thus, all these points are alternate optimal solutions to the problem. If we used a computer to solve this problem, it would identify only one of the corner points of this edge as the optimal solution.

The fact that alternate optimal solutions sometimes occur is really not a problem because this anomaly does not prevent us from finding an optimal solution to the problem. In fact, in Chapter 7, "Goal Programming and Multiple Objective Optimization," you will see that alternate optimal solutions are sometimes very desirable.

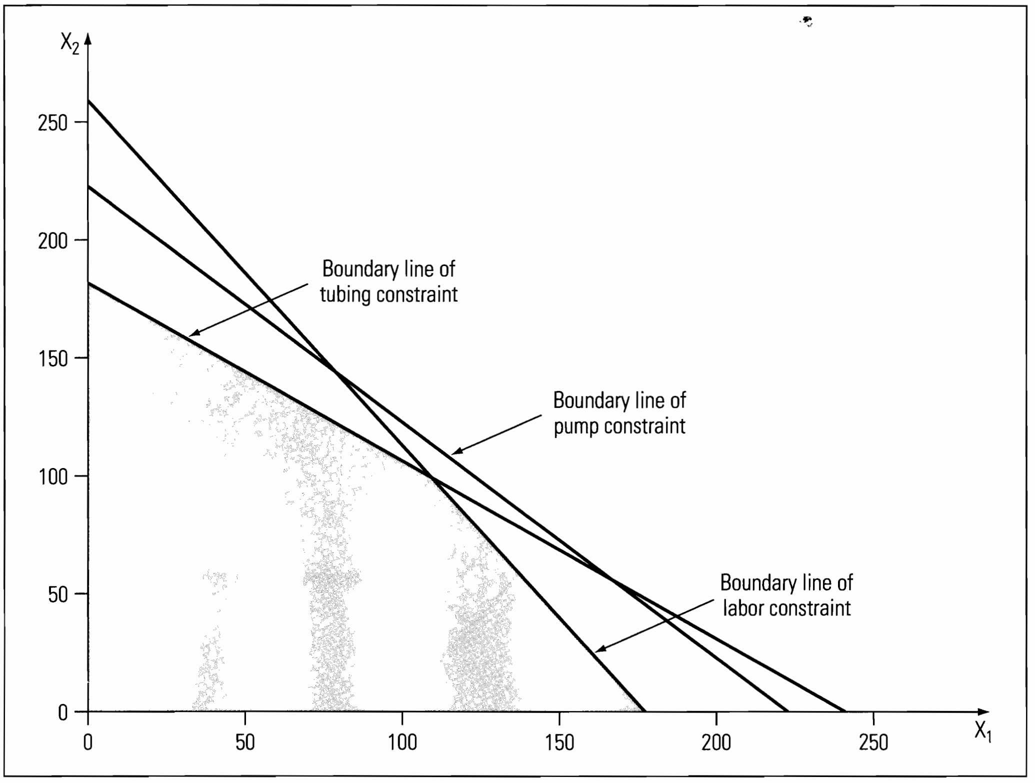
2.11.2 REDUNDANT CONSTRAINTS

Redundant constraints present another special condition that sometimes occurs in an LP model. A **redundant constraint** is a constraint that plays no role in determining the feasible region of the problem. For example, in the hot tub example, suppose that 225 hot tub pumps are available instead of 200. The earlier LP model can be modified as follows to reflect this change:

$$\begin{array}{llcrcl}
\text{MAX:} & 350X_1 & + & 300X_2 & & \\
\text{Subject to:} & 1X_1 & + & 1X_2 & \leq & 225 \\
 & 9X_1 & + & 6X_2 & \leq & 1{,}566 \\
 & 12X_1 & + & 16X_2 & \leq & 2{,}880 \\
 & 1X_1 & & & \geq & 0 \\
 & & & 1X_2 & \geq & 0
\end{array}$$

FIGURE 2.10

Example of a redundant constraint



This model is identical to the original model we formulated for this problem *except* for the new upper limit on the first constraint (representing the number of pumps that can be used). The constraints and feasible region for this revised model are shown in Figure 2.10.

Notice that the pump constraint in this model no longer plays any role in defining the feasible region of the problem. That is, as long as the tubing constraint and labor constraints are satisfied (which is always the case for any feasible solution), then the pump constraint will also be satisfied. Therefore, we can remove the pump constraint from the model without changing the feasible region of the problem—the constraint is simply redundant.

The fact that the pump constraint does not play a role in defining the feasible region in Figure 2.10 implies that there will always be an excess number of pumps available. Because none of the feasible solutions identified in Figure 2.10 fall on the boundary line of the pump constraint, this constraint will always be satisfied as a strict inequality ($1X_1 + 1X_2 < 225$) and never as a strict equality ($1X_1 + 1X_2 = 225$).

Again, redundant constraints are not really a problem. They do not prevent us (or the computer) from finding the optimal solution to an LP problem. However, they do represent "excess baggage" for the computer; so if you know that a constraint is redundant, eliminating it saves the computer this excess work. On the other hand, if the model you are working with will be modified and used repeatedly, it might be best to leave any redundant constraints in the model because they might not be redundant in the future. For example, from Figure 2.3, we know that if the availability of pumps is returned to 200, then the pump constraint again plays an important role in defining the feasible region (and optimal solution) of the problem.

2.11.3 UNBOUNDED SOLUTIONS

When attempting to solve some LP problems, you might encounter situations in which the objective function can be made infinitely large (in the case of a maximization problem) or infinitely small (in the case of a minimization problem). As an example, consider this LP problem:

$$\begin{array}{lrcl} \text{MAX:} & X_1 + X_2 & & \\ \text{Subject to:} & X_1 + X_2 & \geq & 400 \\ & -X_1 + 2X_2 & \leq & 400 \\ & X_1 & \geq & 0 \\ & X_2 & \geq & 0 \end{array}$$

The feasible region and some level curves for this problem are shown in Figure 2.11. From this graph, you can see that as the level curves shift farther and farther away from the origin, the objective function increases. Because the feasible region is not bounded in this direction, you can continue shifting the level curve by an infinite amount and make the objective function infinitely large.

Although it is not unusual to encounter an **unbounded** solution when solving an LP model, such a solution indicates that there is something wrong with the formulation—for example, one or more constraints were omitted from the formulation, or a "less than" constraint was entered erroneously as a "greater than" constraint.

While describing how to find the optimal solution to an LP model by enumerating corner points, we noted that this procedure will not always work if the feasible region for the problem is unbounded. Figure 2.11 provides an example of such a situation. The only extreme points for the feasible region in Figure 2.11 occur at the points (400, 0) and

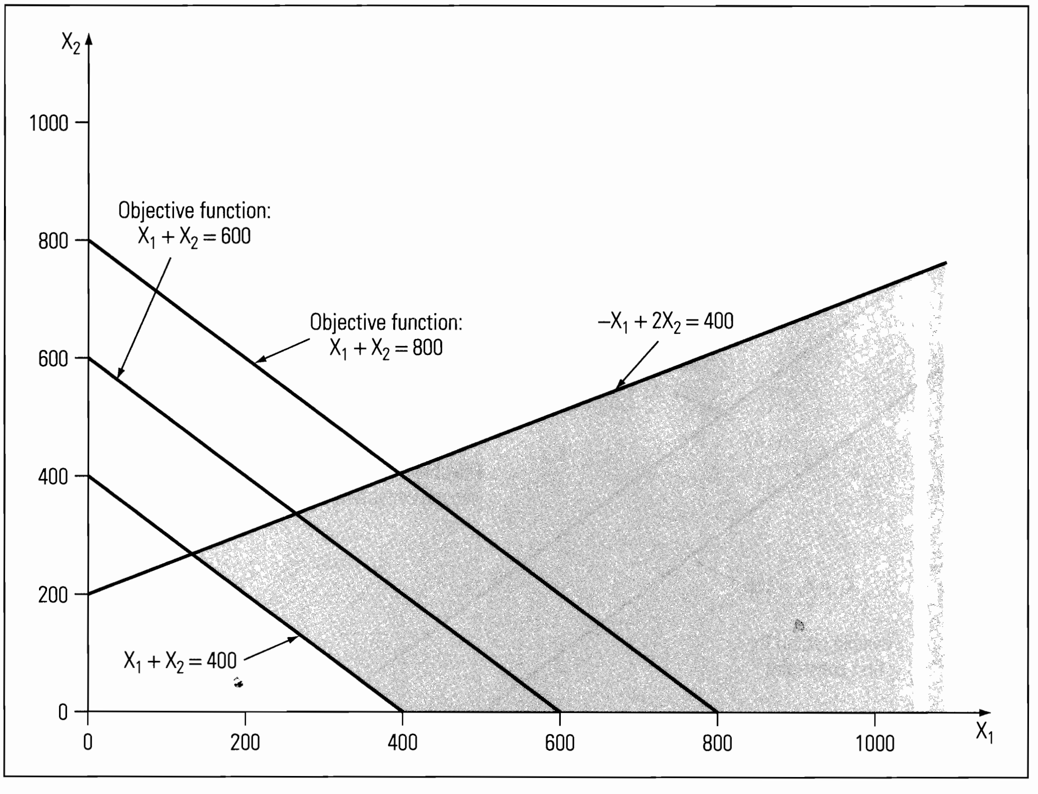


FIGURE 2.11

Example of an LP problem with an unbounded solution

(133.$\bar{3}$, 266.$\bar{6}$). The objective function value at both of these points (and at any point on the line segment joining them) is 400. By enumerating the extreme points for this problem, we might erroneously conclude that alternate optimal solutions to this problem exist that produce an optimal objective function value of 400. This is true if the problem involved *minimizing* the objective function. However, the goal here is to *maximize* the objective function value, which, as we have seen, can be done without limit. So, when trying to solve an LP problem by enumerating the extreme points of an unbounded feasible region, you also must check whether or not the objective function is unbounded.

2.11.4 INFEASIBILITY

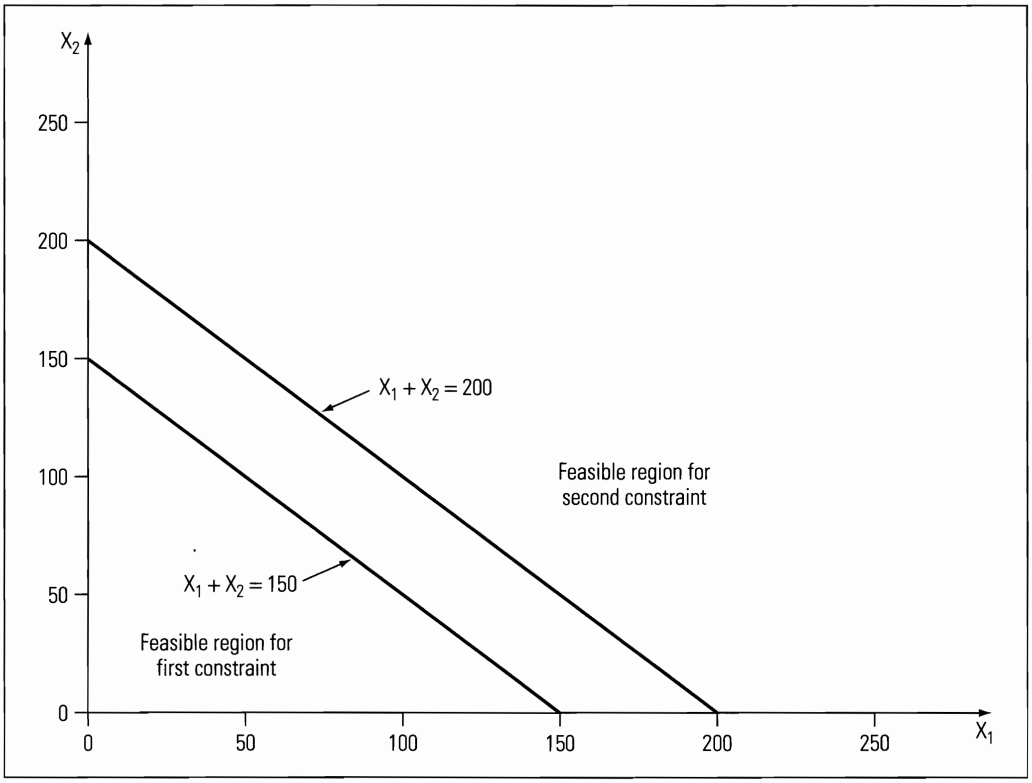
An LP problem is **infeasible** if there is no way to satisfy all the constraints in the problem simultaneously. As an example, consider the LP model:

$$\begin{array}{ll} \text{MAX:} & X_1 + X_2 \\ \text{Subject to:} & X_1 + X_2 \leq 150 \\ & X_1 + X_2 \geq 200 \\ & X_1 \geq 0 \\ & X_2 \geq 0 \end{array}$$

The feasible solutions for the first two constraints in this model are shown in Figure 2.12. Notice that the feasible solutions to the first constraint fall on the left side of its boundary line, whereas the feasible solutions to the second constraint fall on the right side of its boundary line. Therefore, no possible values for X_1 and X_2 exist that satisfy both constraints in the model simultaneously. In such a case, there are no feasible solutions to the problem.

FIGURE 2.12

Example of an LP problem with no feasible solution



Infeasibility can occur in LP problems, perhaps because of an error in the formulation of the model—such as unintentionally making a "less than or equal to" constraint a "greater than or equal to" constraint. Or there just might not be a way to satisfy all the constraints in the model. In this case, constraints would have to be eliminated or loosened to obtain a feasible region (and feasible solution) for the problem.

Loosening constraints involves increasing the upper limits (or reducing the lower limits) to expand the range of feasible solutions. For example, if we loosen the first constraint in the previous model by changing the upper limit from 150 to 250, there is a feasible region for the problem. Of course, loosening constraints should not be done arbitrarily. In a real model, the value 150 would represent some actual characteristic of the decision problem (such as the number of pumps available to make hot tubs). We obviously cannot change this value to 250 unless it is appropriate to do so—that is, unless we know another 100 pumps can be obtained.

2.12 Summary

This chapter provided an introduction to an area of management science known as mathematical programming (MP), or optimization. Optimization covers a broad range of problems that share a common goal—determining the values for the decision variables in a problem that will maximize (or minimize) some objective function while satisfying various constraints. Constraints impose restrictions on the values that can be assumed by the decision variables and define the set of feasible options (or the feasible region) for the problem.

Linear programming (LP) problems represent a special category of MP problems in which the objective function and all the constraints can be expressed as linear combinations of the decision variables. Simple, two-variable LP problems can be solved graphically by identifying the feasible region and plotting level curves for the objective function. An optimal solution to an LP problem always occurs at a corner point of its feasible region (unless the objective function is unbounded).

Some anomalies can occur in optimization problems; these include alternate optimal solutions, redundant constraints, unbounded solutions, and infeasibility.

2.13 References

Bazaraa, M. and J. Jarvis. *Linear Programming and Network Flows*. New York: Wiley, 1990.
Dantzig, G. *Linear Programming and Extensions*. Princeton, NJ: Princeton University Press, 1963.
Eppen, G., F. Gould, and C. Schmidt, *Introduction to Management Science*. Englewood Cliffs, NJ: Prentice Hall, 1993.
Shogan, A. *Management Science*. Englewood Cliffs, NJ: Prentice Hall, 1988.
Winston, W. *Operations Research: Applications and Algorithms*. Belmont, CA: Duxbury Press, 1997.

Questions and Problems

1. An LP model can have more than one optimal solution. Is it possible for an LP model to have exactly two optimal solutions? Why or why not?
2. In the solution to the Blue Ridge Hot Tubs problem, the optimal values for X_1 and X_2 turned out to be integers (whole numbers). Is this a general property of the solutions to LP problems? In other words, will the solution to an LP problem always consist of integers? Why or why not?

3. To determine the feasible region associated with "less than or equal to" constraints or "greater than or equal to" constraints, we graphed these constraints as if they were "equal to" constraints. Why is this possible?
4. Are the following objective functions for an LP model equivalent? That is, if they are both used, one at a time, to solve a problem with exactly the same constraints, will the optimal values for X_1 and X_2 be the same in both cases? Why or why not?

$$\begin{aligned} \text{MAX:} \quad & 2X_1 + 3X_2 \\ \text{MIN:} \quad & -2X_1 - 3X_2 \end{aligned}$$

5. Which of the following constraints are not linear or cannot be included as a constraint in a linear programming problem?
   a. $2X_1 + X_2 - 3X_3 \geq 50$
   b. $2X_1 + \sqrt{X_2} \geq 60$
   c. $4X_1 - \frac{1}{3}X_2 = 75$
   d. $\dfrac{3X_1 + 2X_2 - 3X_3}{X_1 + X_2 + X_3} \leq 0.9$
   e. $3X_1^2 + 7X_2 \leq 45$
6. Solve the following LP problem graphically by enumerating the corner points.

$$\begin{aligned} \text{MAX:} \quad & 3X_1 + 4X_2 \\ \text{Subject to:} \quad & X_1 \leq 12 \\ & X_2 \leq 10 \\ & 4X_1 + 6X_2 \leq 72 \\ & X_1, X_2 \geq 0 \end{aligned}$$

7. Solve the following LP problem graphically using level curves.

$$\begin{aligned} \text{MIN:} \quad & 2X_1 + 3X_2 \\ \text{Subject to:} \quad & 2X_1 + 1X_2 \geq 3 \\ & 4X_1 + 5X_2 \geq 20 \\ & 2X_1 + 8X_2 \geq 16 \\ & 5X_1 + 6X_2 \leq 60 \\ & X_1, X_2 \geq 0 \end{aligned}$$

8. Solve the following LP problem graphically using level curves.

$$\begin{aligned} \text{MAX:} \quad & 2X_1 + 5X_2 \\ \text{Subject to:} \quad & 6X_1 + 5X_2 \leq 60 \\ & 2X_1 + 3X_2 \leq 24 \\ & 3X_1 + 6X_2 \leq 48 \\ & X_1, X_2 \geq 0 \end{aligned}$$

9. Solve the following LP problem graphically by enumerating the corner points.

$$\begin{aligned} \text{MIN:} \quad & 5X_1 + 20X_2 \\ \text{Subject to:} \quad & X_1 + X_2 \geq 12 \\ & 2X_1 + 5X_2 \geq 40 \\ & X_1 + X_2 \leq 15 \\ & X_1, X_2 \geq 0 \end{aligned}$$

10. Consider the following LP problem.

$$\begin{array}{ll} \text{MAX:} & 3X_1 + 2X_2 \\ \text{Subject to:} & 3X_1 + 3X_2 \leq 300 \\ & 6X_1 + 3X_2 \leq 480 \\ & 3X_1 + 3X_2 \leq 480 \\ & X_1, X_2 \geq 0 \end{array}$$

   a. Sketch the feasible region for this model.
   b. What is the optimal solution?
   c. Identify any redundant constraints in this model.

11. Solve the following LP problem graphically by enumerating the corner points.

$$\begin{array}{ll} \text{MAX:} & 10X_1 + 12X_2 \\ \text{Subject to:} & 8X_1 + 6X_2 \leq 98 \\ & 6X_1 + 8X_2 \leq 98 \\ & X_1 + X_2 \geq 14 \\ & X_1, X_2 \geq 0 \end{array}$$

12. Solve the following LP problem using level curves.

$$\begin{array}{ll} \text{MAX:} & 4X_1 + 5X_2 \\ \text{Subject to:} & 2X_1 + 3X_2 \leq 120 \\ & 4X_1 + 3X_2 \leq 140 \\ & X_1 + X_2 \geq 80 \\ & X_1, X_2 \geq 0 \end{array}$$

13. The marketing manager for Mountain Mist soda needs to decide how many TV spots and magazine ads to run during the next quarter. Each TV spot costs $5,000 and is expected to increase sales by 300,000 cans. Each magazine ad costs $2,000 and is expected to increase sales by 500,000 cans. A total of $100,000 may be spent on TV and magazine ads; however, Mountain Mist wants to spend no more than $70,000 on TV spots and no more than $50,000 on magazine ads. Mountain Mist earns a profit of $0.05 on each can it sells.
   a. Formulate an LP model for this problem.
   b. Sketch the feasible region for this model.
   c. Find the optimal solution to the problem using level curves.
14. Blacktop Refining extracts minerals from ore mined at two different sites in Montana. Each ton of ore type 1 contains 20% copper, 20% zinc and 15% magnesium. Each ton of ore type 2 contains 30% copper, 25% zinc and 10% magnesium. Ore type 1 costs $90 per ton and ore type 2 costs $120 per ton. Blacktop would like to buy enough ore to extract at least 8 tons of copper, 6 tons of zinc, and 5 tons of magnesium in the least costly manner.
   a. Formulate an LP model for this problem.
   b. Sketch the feasible region for this problem.
   c. Find the optimal solution.
15. The Electrotech Corporation manufactures two industrial-sized electrical devices: generators and alternators. Both of these products require wiring and testing during the assembly process. Each generator requires 2 hours of wiring and 1 hour of testing and can be sold for a $250 profit. Each alternator requires 3 hours of wiring and 2 hours of testing and can be sold for a $150 profit. There are 260 hours of wiring

time and 140 hours of testing time available in the next production period and Electrotech wants to maximize profit.

a. Formulate an LP model for this problem.
b. Sketch the feasible region for this problem.
c. Determine the optimal solution to this problem using level curves.

16. Refer to the previous question. Suppose that Electrotech's management decides that they need to make at least 20 generators and at least 20 alternators.
    a. Reformulate your LP model to account for this change.
    b. Sketch the feasible region for this problem.
    c. Determine the optimal solution to this problem by enumerating the corner points.
    d. Suppose that Electrotech can acquire additional wiring time at a very favorable cost. Should it do so? Why or why not?
17. Bill's Grill is a popular college restaurant that is famous for its hamburgers. The owner of the restaurant, Bill, mixes fresh ground beef and pork with a secret ingredient to make delicious quarter-pound hamburgers that are advertised as having no more than 25% fat. Bill can buy beef containing 80% meat and 20% fat at $0.85 per pound. He can buy pork containing 70% meat and 30% fat at $0.65 per pound. Bill wants to determine the minimum cost way to blend the beef and pork to make hamburgers that have no more than 25% fat.
    a. Formulate an LP model for this problem. (*Hint*: The decision variables for this problem represent the percentage of beef and the percentage of pork to combine.)
    b. Sketch the feasible region for this problem.
    c. Determine the optimal solution to this problem by enumerating the corner points.
18. Zippy motorcycle manufacturing produces two popular pocket bikes (miniature motorcycles with 49cc engines): the Razor and the Zoomer. In the coming week, the manufacturer wants to produce a total of up to 700 bikes and wants to ensure that the number of Razors produced does not exceed the number of Zoomers by more than 300. Each Razor produced and sold results in a profit of $70, and each Zoomer results in a profit of $40. The bikes are identical mechanically and differ only in the appearance of the polymer-based trim around the fuel tank and seat. Each Razor's trim requires 2 pounds of polymer and 3 hours of production time, and each Zoomer requires 1 pound of polymer and 4 hours of production time. Assume that 900 pounds of polymer and 2400 labor hours are available for production of these items in the coming week.
    a. Formulate an LP model for this problem.
    b. Sketch the feasible region for this problem.
    c. What is the optimal solution?
19. The Quality Desk Company makes two types of computer desks from laminated particle board. The Presidential model requires 30 square feet of particle board, 1 keyboard sliding mechanism, and 5 hours of labor to fabricate. It sells for $149. The Senator model requires 24 square feet of particle board, 1 keyboard sliding mechanism, and 3 hours of labor to fabricate. It sells for $135. In the coming week, the company can buy up to 15,000 square feet of particle board at $1.35 per square foot and up to 600 keyboard sliding mechanisms at a cost of $4.75 each. The company views manufacturing labor as a fixed cost and has 3000 labor hours available in the coming week for the fabrication of these desks.
    a. Formulate an LP model for this problem.
    b. Sketch the feasible region for this problem.
    c. What is the optimal solution?
20. A farmer in Georgia has a 100-acre farm on which to plant watermelons and cantaloupes. Every acre planted with watermelons requires 50 gallons of water per day and must be prepared for planting with 20 pounds of fertilizer. Every acre planted

with cantaloupes requires 75 gallons of water per day and must be prepared for planting with 15 pounds of fertilizer. The farmer estimates that it will take 2 hours of labor to harvest each acre planted with watermelons and 2.5 hours to harvest each acre planted with cantaloupes. He believes that watermelons will sell for about $3 each, and cantaloupes will sell for about $1 each. Every acre planted with watermelons is expected to yield 90 salable units. Every acre planted with cantaloupes is expected to yield 300 salable units. The farmer can pump about 6,000 gallons of water per day for irrigation purposes from a shallow well. He can buy as much fertilizer as he needs at a cost of $10 per 50-pound bag. Finally, the farmer can hire laborers to harvest the fields at a rate of $5 per hour. If the farmer sells all the watermelons and cantaloupes he produces, how many acres of each crop should the farmer plant to maximize profits?

a. Formulate an LP model for this problem.
b. Sketch the feasible region for this model.
c. Find the optimal solution to the problem using level curves.

21. Sanderson Manufacturing produces ornate, decorative wood frame doors and windows. Each item produced goes through 3 manufacturing processes: cutting, sanding, and finishing. Each door produced requires 1 hour in cutting, 30 minutes in sanding, and 30 minutes in finishing. Each window requires 30 minutes in cutting, 45 minutes in sanding, and 1 hour in finishing. In the coming week Sanderson has 40 hours of cutting capacity available, 40 hours of sanding capacity, and 60 hours of finishing capacity. Assume that all doors produced can be sold for a profit of $500 and all windows can be sold for a profit of $400.
    a. Formulate an LP model for this problem.
    b. Sketch the feasible region.
    c. What is the optimal solution?
22. PC-Express is a computer retail store that sells two kinds of microcomputers: desktops and laptops. The company earns $600 on each desktop computer it sells and $900 on each laptop. The microcomputers PC-Express sells are manufactured by another company. This manufacturer has a special order to fill for another customer and cannot ship more than 80 desktop computers and 75 laptops to PC-Express next month. The employees at PC-Express must spend about 2 hours installing software and checking each desktop computer they sell. They spend roughly 3 hours to complete this process for laptop computers. They expect to have about 300 hours available for this purpose during the next month. The store's management is fairly certain that they can sell all the computers they order, but are unsure how many desktops and laptops they should order to maximize profits.
    a. Formulate an LP model for this problem.
    b. Sketch the feasible region for this model.
    c. Find the optimal solution to the problem by enumerating the corner points.
23. American Auto is evaluating their marketing plan for the sedans, SUVs, and trucks they produce. A TV ad featuring this SUV has been developed. The company estimates that each showing of this commercial will cost $500,000 and increase sales of SUVs by 3%, but reduce sales of trucks by 1%, and have no effect of the sales of sedans. The company also has a print ad campaign developed that it can run in various nationally distributed magazines at a cost of $750,000 per title. It is estimated that each magazine title the ad runs in will increase the sales of sedans, SUVs, and trucks by 2%, 1%, and 4%, respectively. The company desires to increase sales of sedans, SUVs, and trucks by at least 3%, 14%, and 4%, respectively, in the least costly manner.
    a. Formulate an LP model for this problem.
    b. Sketch the feasible region.
    c. What is the optimal solution?

CASE 2.1

For the Lines They Are A-Changin' (with apologies to Bob Dylan)

The owner of Blue Ridge Hot Tubs, Howie Jones, has asked for your assistance in analyzing how the feasible region and solution to his production problem might change in response to changes in various parameters in the LP model. He is hoping that this might further his understanding of LP and how the constraints, objective function, and optimal solution interrelate. To assist in this process, he asked a consulting firm to develop the spreadsheet shown earlier in Figure 2.8 (and the file Fig. 2-8.xls on your data disk) that dynamically updates the feasible region and optimal solution and the various parameters in the model change. Unfortunately, Howie has not had much time to play around with this spreadsheet, so he has left it in your hands and asked you to use it to answer the following questions. (Click the Reset button in file Fig. 2-8.xls before answering each of the following questions.)

Note: The file Fig2-8.xls contains a macro that must be enabled for the workbook to operate correctly. To allow this (and other) macros to run in Excel click: Office button, Excel options, Trust Center, Trust Center Settings, Macro Settings, select "Disable all macros with notification", click OK. Then when Excel opens a workbook containing macros it will display a security warning indicating some active content has been disabled and will give you the opportunity to enable this content, which you should do for the Excel files accompanying this book.

a. In the optimal solution to this problem, how many pumps, hours of labor, and feet of tubing are being used?
b. If the company could increase the number of pumps available, should they? Why or why not? And if so, what is the maximum number of additional pumps that they should consider acquiring, and by how much would this increase profit?
c. If the company could acquire more labor hours, should they? Why or why not? If so, how much additional labor should they consider acquiring and by how much would this increase profit?
d. If the company could acquire more tubing, should they? Why or why not? If so, how additional tubing should they consider acquiring and how much would this increase profit?
e. By how much would profit increase if the company could reduce the labor required to produce Aqua-Spas from 9 to 8 hours? And from 8 to 7 hours? And from 7 to 6 hours?
f. By how much would profit increase if the company could reduce the labor required to produce Hydro-Luxes from 6 to 5 hours? And from 5 to 4 hours? And from 4 to 3 hours?
g. By how much would profit increase if the company could reduce the amount of tubing required to produce Aqua-Spas from 12 to 11 feet? And from 11 to 10 feet? And from 10 to 9 feet?
h. By how much would profit increase if the company could reduce the amount of tubing required to produce Hydro-Luxes from 16 to 15 feet? And from 15 to 14 feet? And from 14 to 13 feet?
i. By how much would the unit profit on Aqua-Spas have to change before the optimal product mix changes?
j. By how much would the unit profit on Hydro-Luxes have to change before the optimal product mix changes?